A note on Morse's index theorem for Perelman's $\mathcal{L}$-length

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Abstract This is essentially a note on Section 7 of Perelman's first paper on Ricci flow. We list some basic properties of the index form for Perelman's $\mathcal{L}$-length, which are analogous to the ones in Riemannian case (with fixed metric), and observe that Morse's index theorem for Perelman's $\mathcal{L}$-length holds. As a corollary we get the finiteness of the number of the $\mathcal{L}$-conjugate points along a finite $\mathcal{L}$-geodesic.

In his ground-breaking work [6] on Ricci flow Perelman introduced $\mathcal{L}$-length, $\mathcal{L}$-Jacobi field among many other important innovations. For more details see [2],[3], and [5]. Here we'll add some notes on Section 7 of this paper of Perelman's. We list some basic properties of the index form for Perelman's $\mathcal{L}$-length, which are analogous to the ones in Riemannian case (with fixed metric, cf.[1], [4] and [7]), and observe that Morse's index theorem for Perelman's $\mathcal{L}$-length holds. The main idea of the proof is the same as that of the fixed metric case, but one needs to be careful when the τ-interval is $[0,\bar{\tau}]$ (see in particular the proof of the Key Lemma below). As a corollary we get the finiteness of number of the $\mathcal{L}$-conjugate points along a finite $\mathcal{L}$-geodesic.

Throughout this note we assume $(M, g(\tau))$, where $(g_{ij})_\tau = 2R_{ij}$, is a (backwards) Ricci flow, which is complete for each τ-slice and has uniformly bounded curvature operator on an interval $[\tau_1, \tau_2]$.

Recall Perelman's $\mathcal{L}$-length $\mathcal{L}(\gamma)=\int_{\tau_1}^{\tau_2}\sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2)d\tau$ for a curve $\gamma(\tau)$ $(\tau_1 \le \tau \le \tau_2)$ in M.

Definition 1 Let $p \in M$, $v \in T_pM$, and γ_v be the $\mathcal{L}$-geodesic with $\gamma_v(0) = p$, $lim_{\tau\to 0}\sqrt{\tau}\dot{\gamma}_v(\tau) = v$. We say $q = \gamma_v(\bar{\tau})$ is a $\mathcal{L}$-conjugate point of p along the $\mathcal{L}$-geodesic γ_v if v is a critical point of the $\mathcal{L}_{\bar{\tau}}$-exponential map $\mathcal{L}_{\bar{\tau}}exp$. Here, (following the notation in [2],) $\mathcal{L}_{\bar{\tau}}exp(v)=\mathcal{L}exp_v(\bar{\tau})$ $(=\gamma_v(\bar{\tau}))$.

Definition 2(Perelman) A vector field along a $\mathcal{L}$-geodesic γ is called $\mathcal{L}$-Jacobi field, if it is the variation vector field of a one parameter family of $\mathcal{L}$-geodesics γ_s with $\gamma_0 = \gamma$.

The equation for a $\mathcal{L}$-Jacobi field U along a $\mathcal{L}$-geodesic $\gamma_v(\tau)$ ($0 < \tau_1 \le \tau \le \tau_2$) is (see, for example, [2])

$\nabla_X\nabla_X U - R(X,U)X - 1/2\nabla_U(\nabla R) + 2(\nabla_U \mathrm{Ric})(X) + 2\mathrm{Ric}(\nabla_X U) + 1/(2\tau)\nabla_X U = 0.$

(Here and below, $X(\tau) = \dot{\gamma}_v(\tau)$. Moreover $Ric(Y)$ here means $Ric(Y,\cdot)$ in Perelman [6].)

One can easily extend this to the case $\tau_1 = 0$. (See [2].)

Remark 1 As in the Riemannian case (with fixed metric) $q = \gamma_v(\bar{\tau})$ is a $\mathcal{L}$-conjugate point of $p = \gamma_v(0)$ along the $\mathcal{L}$-geodesic $\gamma_v(\tau)(0 \le \tau \le \bar{\tau})$ if and only if there is a nontrivial $\mathcal{L}$-Jacobi field U along γ_v with $U(0) = U(\bar{\tau}) = 0$.

The $\mathcal{L}$-index form along a $\mathcal{L}$-geodesic $\gamma_v(\tau)$ ($\tau_1 \le \tau \le \tau_2$) is defined to be

$I(U,V) = \int_{\tau_1}^{\tau_2} \tau^{1/2}[(\mathrm{Hess}R)(U,V)+2\langle \nabla_X U, \nabla_X V\rangle + 2\langle R(U,X)V,X\rangle - 2(\nabla_U \mathrm{Ric})(V,X) - 2(\nabla_V \mathrm{Ric})(U,X) + 2(\nabla_X \mathrm{Ric})(U,V)]d\tau$

for any piecewise smooth vector fields U,V along γ_v.

This is a symmetric, bilinear form.

Remark 2 If $Y(\tau)$ is a smooth vector field along a $\mathcal{L}$-geodesic $\gamma_v(\tau)(0 \le \tau \le \bar{\tau})$, and $Y(0)=0$, then $I(Y,Y) = \delta_Y^2 \mathcal{L} - \delta_{\nabla_Y Y}\mathcal{L}$. (Compare with formula (7.7) in Perelman [6].)

Now we prove a

Key Lemma For any vector fields U,V along a $\mathcal{L}$-geodesic $\gamma_v(\tau)(0 < \tau_1 \le \tau \le \tau_2)$ with U smooth (and V piecewise smooth), we have

$I(U,V) = 2\tau^{1/2}\langle \nabla_X U, V\rangle|_{\tau_1}^{\tau_2} - 2\int_{\tau_1}^{\tau_2} \tau^{1/2}\langle \nabla_X\nabla_X U - R(X,U)X - 1/2\nabla_U(\nabla R) + 2(\nabla_U \mathrm{Ric})(X) + 2\mathrm{Ric}(\nabla_X U) + 1/(2\tau)\nabla_X U, V\rangle d\tau$.

Furthermore, this equality extends to the case $\tau_1 = 0$.

Proof In case $\tau_1 > 0$ one simply use

$d/d\tau \langle \nabla_X U, V\rangle = \langle \nabla_X\nabla_X U, V\rangle + \langle \nabla_X U, \nabla_X V\rangle + 2\mathrm{Ric}(\nabla_X U, V) + (\nabla_X \mathrm{Ric})(U,V) + (\nabla_U \mathrm{Ric})(V,X) - (\nabla_V \mathrm{Ric})(X,U)$

(compare with formula (11.2) in [3]), and

$d/d\tau(\tau^{1/2}\langle \nabla_X U, V\rangle) = (1/2)\tau^{-1/2}\langle \nabla_X U, V\rangle + \tau^{1/2} d/d\tau \langle \nabla_X U, V\rangle$,

then integration by parts, and the formula follows.

To justify the $\tau_1 = 0$ case, it suffices to observe

$\tau(\nabla_X\nabla_X U - R(X,U)X - 1/2\nabla_U(\nabla R) + 2(\nabla_U \mathrm{Ric})(X) + 2\mathrm{Ric}(\nabla_X U) + 1/(2\tau)\nabla_X U) = \nabla_{\sqrt{\tau}X}\nabla_{\sqrt{\tau}X} U - R(\sqrt{\tau}X, U)\sqrt{\tau}X - \tau/2\nabla_U(\nabla R) + 2\sqrt{\tau}(\nabla_U \mathrm{Ric})(\sqrt{\tau}X) + 2\sqrt{\tau}\mathrm{Ric}(\nabla_{\sqrt{\tau}X} U)$

(compare with [2]), and note that $lim_{\tau\to 0}\sqrt{\tau}X(\tau)$ exists, and that the integration $\int_0^1 \tau^{-1/2} d\tau$ converges.

Remark 3 One can easily generalize the Key Lemma to the case that U is piecewise smooth.

Below we list some basic properties of $\mathcal{L}$-Jacobi field which is analogous to the Riemannian case (with fixed metric, see [1], [4], and in particular [7]), whose proof is similar to the fixed metric case and is omitted (in the proof the Key Lemma above play an important role).

For convenience we denote by $\mathcal{V}_0(\tau_1,\tau_2)$ the space of piecewise smooth vector fields $V(\tau)$ along a $\mathcal{L}$-geodesic $\gamma_v(\tau)$ ($\tau_1 \le \tau \le \tau_2$) with $V(\tau_1) = V(\tau_2) = 0$.

In the following lemmata we suppose $\gamma_v(\tau)$ ($\tau_1 \le \tau \le \tau_2$) is a $\mathcal{L}$-geodesic.

Lemma 1 Let $\gamma_v(\tau_2)$ be $\mathcal{L}$-conjugate to $\gamma_v(\tau_1)$ along γ_v. Then for any $\mathcal{L}$-Jacobi field $U \in \mathcal{V}_0(\tau_1,\tau_2)$ one has $I(U,U) = 0$.

Lemma 2 If $\gamma_v(\tau)$($\tau_1 \le \tau \le \tau_2$) does not contain any $\mathcal{L}$-conjugate point of $\gamma_v(\tau_1)$, then the $\mathcal{L}$-index form is positive definite on $\mathcal{V}_0(\tau_1,\tau_2)$.

Lemma 3 Let $\gamma_v(\tau_2)$ be $\mathcal{L}$-conjugate to $\gamma_v(\tau_1)$ along γ_v, but for any τ such that $\tau_1 < \tau < \tau_2$, $\gamma_v(\tau)$ is not $\mathcal{L}$-conjugate to $\gamma_v(\tau_1)$ along γ_v. Then the $\mathcal{L}$-index form is positive semi-definite (but not positive definite) on $\mathcal{V}_0(\tau_1, \tau_2)$.

Lemma 4 There exists τ' with $\tau_1 < \tau' < \tau_2$ such that $\gamma_v(\tau')$ is $\mathcal{L}$-conjugate to $\gamma_v(\tau_1)$ along γ_v if and only if there exists a vector field $Y \in \mathcal{V}_0(\tau_1, \tau_2)$ such that $I(Y, Y) < 0$.

Lemma 5 U is a $\mathcal{L}$-Jacobi field if and only if $I(U, Y) = 0$ for any vector field $Y \in \mathcal{V}_0(\tau_1, \tau_2)$.

Lemma 6 Suppose $\gamma_v(\tau)$ does not contain any $\mathcal{L}$-conjugate point of $\gamma_v(\tau_1)$. Let U, Y be piecewise smooth vector field along $\gamma_v(\tau)$ with $U(\tau_1) = Y(\tau_1)$, $U(\tau_2) = Y(\tau_2)$, and U is a $\mathcal{L}$-Jacobi field. Then $I(U, U) \leq I(Y, Y)$. The equality holds if and only if $Y = U$.

Remark 4 Lemma 6 was used in Perelman [6](7.11).

Lemma 7 Suppose that $\gamma_v(\tau_2)$ is not $\mathcal{L}$-conjugate to $\gamma_v(\tau_1)$ along γ_v. Then given any $w \in T_{\gamma_v(\tau_2)}M$ there exists an unique $\mathcal{L}$-Jacobi field U along γ_v such that $U(\tau_1) = 0$ and $U(\tau_2) = w$.

Let $\gamma_v(\tau)(0 \leq \tau \leq \bar{\tau})$ be a $\mathcal{L}$-geodesic. The index of the $\mathcal{L}$-index form I along γ_v is defined to be the maximum dimension of a subspace of $\mathcal{V}_0(0, \bar{\tau})$ on which I is negative definite.

Now we can state Morse's index theorem for Perelman's $\mathcal{L}$-length.

Theorem The index of $\mathcal{L}$-index form along a $\mathcal{L}$-geodesic $\gamma_v(\tau)(0 \leq \tau \leq \bar{\tau})$ is equal to the number (counting with multiplicity) of $\mathcal{L}$-conjugate points $\gamma_v(\tau')(0 < \tau' < \bar{\tau})$ of $\gamma_v(0)$ along γ_v. The index is always finite.

Here, by definition, the multiplicity of a $\mathcal{L}$-conjugate point $\gamma_v(\tau')$ of $\gamma_v(0)$ along a $\mathcal{L}$-geodesic γ_v is the dimension of subspace that consists of all $\mathcal{L}$-Jacobi fields in $\mathcal{V}_0(0, \tau')$.

Proof As in the fixed metric case, the main idea is trying to reduce the problem to a finite dimensional subspace of $\mathcal{V}_0(0, \bar{\tau})$, using the lemmata above. The detail is similar to that of the fixed metric case (see [1], [4] and [7]) and is omitted.

We have the following immediate

Corollary The number of the $\mathcal{L}$-conjugate points of $\gamma_v(0)$ along $\gamma_v(\tau)(0 \leq \tau \leq \bar{\tau})$ is finite.

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